UK University Integration Bee Basic Techniques Guide

A guide covering the foundations of integration technique

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$$\int$$

Contents

Introduction

This booklet is intended to reinforce the basic techniques of integration - it's assumed you know the integration covered at A level - integration by substitution and by parts. Despite being quite basic, these techniques can take you a long way and form the foundation of any more advanced method. They can be used to evaluate a lot of interesting integrals, derive results of theoretical importance and to us are interesting in themselves - it's really satisfying being able to manage a problem with clever substitutions and algebraic manipulations over using a complicated and powerful technique like contour integration.

It's also intended as a support to the UK University Integration Bee (website can be found here) which is run across several universities in the UK - a team integration competition. The techniques in here are sufficient to solve a lot of the problems but not all - later this booklet will be developed into a book. This book will cover the major methods of integration and theory such as special functions which will be applied to derive results of importance in other areas of mathematics. Particularly theoretical areas which aren't of much importance if you just want to learn techniques for integrating will be marked with a *.

We hope that this booklet and the competition will help convince you that these integration techniques are more than just tricks but that these integrals are interesting problems which are also useful!

1 Basic Techniques

Life's difficulties do not come in increasing order.

– Neshan Wickramasekera

1.1 Odd and Even functions

An odd function is one such that $f(x) = -f(-x)$ and an even function is one for which $f(x) = f(-x)$. Examples of odd functions are x, $\sin(x)$ and $\tan(x)$ while examples of even functions are $|x|$, $\cos(x)$ and x^2. The symmetry they have make evaluating certain integrals easier for example if you were confronted with evaluating the following integral

$$\int_{-1}^{1} x^3 \mathrm{d}x \tag{1.1}$$

then the usual method would be to go ahead and directly calculate it:

$$\int_{-1}^{1} x^3 \mathrm{d}x = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = 0$$

Or you could notice that, because $f(x) = x^3$ is an odd function that the two halves of the integral - from $[-1, 0]$ and $[0, 1]$ cancel out

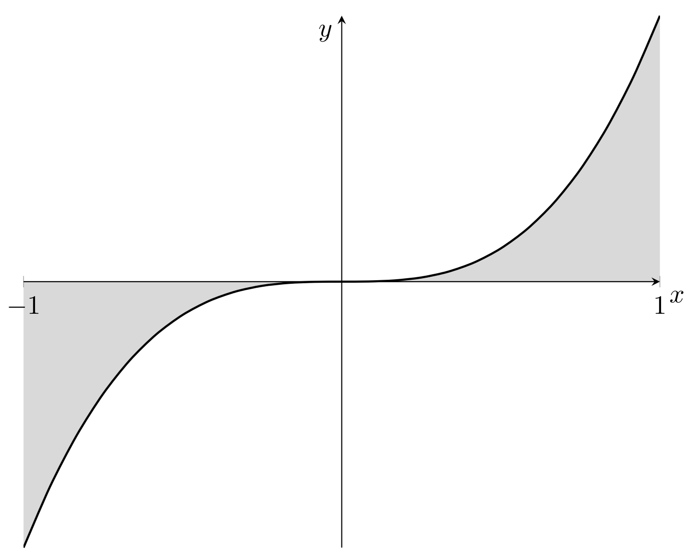

Figure 1: x^3 on the interval $[-1, 1]$. The two shaded areas are the same but opposite sign.

so we can immediately see that the integral is 0. In general, we have that, for an odd function f

$$\int_{-a}^{a} f(x)\mathrm{d}x = 0 \quad \forall a \in \mathbb{R} \tag{1.2}$$

The counterpart for even functions isn't quite as dramatic - the symmetry in this case is that on the two halves the integrals are equal so instead we have the result

$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x \tag{1.3}$$

While this doesn't seem as useful now, it can simplify integrals a lot.

For a more extreme example of these techniques, consider the integral

$$\int_{-2}^{2} \left(x^3 \cos\left(\frac{x}{2}\right) + \frac{1}{2}\right)\sqrt{4-x^2}\mathrm{d}x \tag{1.4}$$

This integral is also well known from some meme about solving this for a wifi password

Figure 2: The integral you're faced with when trying to access wifi at Nanjing University of Aeronautics and Astronautics

It looks daunting at first - maybe not even worth the effort for the wifi - but with the previous idea, it's not bad:

$$I_1 = \int_{-2}^{2} x^3 \cos\left(\frac{x}{2}\right)\sqrt{4-x^2}\mathrm{d}x \;\&\; I_2 = \frac{1}{2}\int_{-2}^{2}\sqrt{4-x^2}\mathrm{d}x$$

The first integral is the nasty part but after squinting at it for long enough, you can see that the integrand is an odd function - both the cosine and square root are even while x^3 is odd so $I_1 = 0$. Squinting at I_2, you see that the integrand is an even function so we can simplify it as

$$I_2 = \int_0^2 \sqrt{4 - x^2}\mathrm{d}x$$

Now this integral (and before this simplification) can be done with a trigonometric substitution but a neater and more geometric way of viewing the problem is noting that if $y = \sqrt{4 - x^2}$ then $x^2 + y^2 = 4$ i.e the equation of a circle. Those bounds in particular represent a quarter of a circle of radius 2 so that the final answer is just π.

Sometimes this trick can really help save time like in the 2021 Shuttle where being able to answer the question as fast as possible was really useful:

$$I = \int_{-1}^{1} \frac{\sin(\cot^{-1} x) + \cos(\tan^{-1} x)}{x^2 + 1}\mathrm{d}x \tag{1.5}$$

While you can work out both of the expressions on the numerator without too much difficulty, some time can be saved by noticing that the sine term is entirely an odd function so its contribution to the integral is 0. So you can work with only the cosine term. To work out $\cos(\tan^{-1}(x))$, it's useful to use a triangle

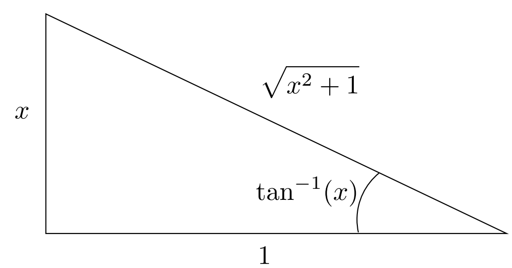

Figure 3: A triangle with angle $\tan^{-1}(x)$

From the picture, we see that $\cos(\tan^{-1} x) = \frac{1}{\sqrt{x^2+1}}$ so that

$$I = \int_{-1}^{1} \frac{\mathrm{d}x}{(x^2 + 1)^{\frac{3}{2}}} = \left[\frac{x}{\sqrt{x^2 + 1}}\right]_{-1}^{1} = \sqrt{2}$$

In the competition, the team had to pass on the answer squared minus 2 which in this case was 0. That was then used in the bounds of the next integral - it was from 0 to b so it actually turned out to be 0 by default!

For the next example let's do something fairly general to demonstrate the importance of odd and even decomposition.

The crux of the idea is realising that any function $f(x)$ can be written as a sum of odd and even "parts"

$$f(x) = \frac{1}{2}(\underbrace{(f(x) + f(-x))}_{\text{Even}} + \underbrace{(f(x) - f(-x))}_{\text{Odd}})$$

(if this looks weird, don't worry the main application of it involves just subbing $-x$ into an integral and then adding the two different forms)

Consider now the integral

$$\int_{-a}^{a} \frac{x^2}{e^{\sin x} + 1} \mathrm{d}x \tag{1.6}$$

We can break it down as above, but you might wonder how we could think of this (if it wasn't in the section named odd and even functions...). Clearly there's no right answer to this but the main indicators to me are the bounds of integration, and the fact that $\sin x$ and x^2 have not much to do with each other except for being of opposite parity

Thus we decompose the function as above to get

$$\begin{aligned}
I &= \frac{1}{2}\int_{-a}^{a} \frac{x^2}{e^{\sin x} + 1} + \frac{(-x)^2}{e^{\sin(-x)} + 1}\mathrm{d}x \\
&= \frac{1}{2}\int_{-a}^{a} \frac{x^2(e^{-\sin x} + 1) + x^2(e^{\sin x} + 1)}{(e^{\sin x} + 1)(e^{-\sin x} + 1)}\mathrm{d}x \\
&= \frac{1}{2}\int_{-a}^{a} \frac{x^2(e^{-\sin x} + e^{\sin x} + 2)}{1 + e^{\sin x} + e^{-\sin x} + \underbrace{e^{\sin x}e^{-\sin x}}_{1}}\mathrm{d}x \\
&= \frac{1}{2}\int_{-a}^{a} x^2 \mathrm{d}x = \frac{2a^3}{3}
\end{aligned}$$

As an exercise you could consider the properties of x^2 and $e^{\sin x}$ that made this work and try to generalise it!

1.2 Reflection Substitutions

The previous section shows how powerful the symmetry given by an odd or even function is but that's a very specific situation. The reflection substitution takes advantage of similar symmetries but around points other than the origin.

The actual identity is

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x \tag{1.7}$$

which can be verified by the substitution $u = a + b - x$ but can be viewed geometrically as a reflection in the line $x = \frac{a+b}{2}$. So the odd and even function symmetry uses a reflection substitution too - with $a = -b$. Another name for this type of substitution is King's Substitution but I feel like reflection works better as it highlights one of the key points - we preserve the endpoints by flipping them.

A reflection substitution which turns out to be really useful for trigonometric functions is $u = \frac{\pi}{2} - x$ - taking advantage of the fact that $\sin(\frac{\pi}{2} - x) = \cos(x)$, $\cos(\frac{\pi}{2} - x) = \sin(x)$ and $\tan(\frac{\pi}{2} - x) = \cot(x)$ which can be seen from any right angled triangle.

To demonstrate its use, consider this integral - Putnam 1980 A3:

$$I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\tan^{\alpha}(x)} \tag{1.8}$$

where α is any real number. Since α can be anything, this integral looks really daunting but it also suggests there isn't really much we can actually do - it limits us quite a lot. So a reflection substitution is all we can really do - with $u = \frac{\pi}{2} - x$ we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}u}{1+\frac{1}{\tan^{\alpha}}(u)} = \int_0^{\frac{\pi}{2}} \frac{\tan^{\alpha}(u)}{1+\tan^{\alpha}(u)}\mathrm{d}u$$

The integral has ended up in a very similar looking form - this is what usually happens with reflection substitutions - we get something of a similar form which we can then manipulate alongside our original integral. In this case, if we add the two integrals together, we get

$$2I = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}$$

so that $I = \dfrac{\pi}{4}$. Not so bad for a Putnam problem! In the original problem $\alpha = \sqrt{2}$.

Another classic integral with a more involved use of the reflection substitution is the log sine integral - one of my favourite integrals (there'll be a section on it in the full book!) Consider the integral

$$I = \int_0^{\frac{\pi}{2}} \log(\sin x) \mathrm{d}x \tag{1.9}$$

With the reflection substitution $u = \frac{\pi}{2} - x$ we get

$$I = \int_0^{\frac{\pi}{2}} \log(\cos(x)) \mathrm{d}x$$

A consequence of this is

$$\int_0^{\frac{\pi}{2}} \log(\tan x) \mathrm{d}x = 0$$

Adding these two together, we get

$$\begin{aligned} 2I &= \int_0^{\frac{\pi}{2}} \log(\sin(x)\cos(x)) \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \log(\sin(2x)) - \log(2) \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} \log(\sin(2x)) \mathrm{d}x - \frac{\pi \log 2}{2} \end{aligned}$$

In the integral, substitute $u = 2x$. Then $\mathrm{d}u = 2\mathrm{d}x$ and the bounds become from 0 to π so the integral becomes

$$\frac{1}{2} \int_0^{\pi} \log(\sin(u)) \mathrm{d}u$$

Thinking about the values $\sin(x)$ takes on the interval $[\frac{\pi}{2}, \pi]$, it's the same as the interval $[0, \pi]$ so that

$$\int_0^{\frac{\pi}{2}} \log(\sin(u)) \mathrm{d}u = \int_{\frac{\pi}{2}}^{\pi} \log(\sin(u)) \mathrm{d}u$$

Alternatively we could also use the substitution $u = \pi - x$ and use the fact that $\sin(\pi - x) = \sin(x)$. This means that the above becomes

$$2I = I - \frac{\pi \log 2}{2}$$

which gives us $I = -\dfrac{\pi \log 2}{2}$.

Another problem using the reflection substitution on the way is this one from JEE Advanced 2019:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos\theta}}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^5} \mathrm{d}\theta$$

Beginning with a reflection substitution we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\theta}}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^5} \mathrm{d}\theta \tag{1.10}$$

Adding these two together, we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{(\sqrt{\sin\theta} + \sqrt{\cos\theta})^4} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\cos^2\theta(\sqrt{\tan\theta} + 1)^4}$$

where factoring out the $\cos^2\theta$ term lets us substitute $u = \tan\theta$ so that $\mathrm{d}u = \sec^2\theta \mathrm{d}\theta$. The bounds then become from 0 to ∞ and we get

$$2I = \int_0^{\infty} \frac{\mathrm{d}u}{(1 + \sqrt{u})^4}$$

Now we can substitute $t = \sqrt{u}$ which gives us $\mathrm{d}t = \dfrac{\mathrm{d}u}{2\sqrt{u}} = \dfrac{\mathrm{d}u}{2t}$. This gives

$$I = \int_0^{\infty} \frac{t}{(1+t)^4} \mathrm{d}t \overset{(u=t+1)}{=} \int_1^{\infty} \frac{u-1}{u^4} \mathrm{d}u = \left[-\frac{1}{2}u^{-2} + \frac{1}{3}u^{-3} \right]_1^{\infty} = \frac{1}{6}$$

The substitution $u = \frac{1}{x}$ pairs really well with functions for which $f(\frac{1}{x})$ and $f(x)$ have a close relation. Examples are $\log(\frac{1}{x}) = -\log(x)$ and $\arctan(\frac{1}{x}) = \frac{\pi}{2} - \arctan(x)$ for $x > 0$. In view of the actual substitution made, we get $\mathrm{d}x = -\frac{1}{u^2}\mathrm{d}u$ which works well with an integrand involving $\frac{1}{x^2+1}$ because it becomes

$$\frac{1}{\frac{1}{u^2} + 1} \times -\frac{1}{u^2} \mathrm{d}u = -\frac{1}{u^2 + 1} \mathrm{d}u$$

which is the same form. A good example is

$$I = \int_0^{\infty} \frac{\ln(x)}{x^2 + 1} \mathrm{d}x \tag{1.11}$$

Substituting $u = \frac{1}{x}$ we get, using the above, noting that the boundaries flip giving a minus sign

$$\int_{\infty}^{0} -\frac{\ln(\frac{1}{u})}{u^2 + 1} \mathrm{d}u = \int_0^{\infty} \frac{-\ln(u)}{u^2 + 1} \mathrm{d}u = -I$$

Since $I = -I$, we have $I = 0$. In fact the same argument shows that

$$\int_0^1 \frac{\ln(x)}{x^2 + 1} \mathrm{d}x = -\int_1^{\infty} \frac{\ln(x)}{x^2 + 1} \mathrm{d}x$$

Splitting intervals into parts and identifying results like the above is fairly common. Later on, we'll tackle these individual integrals too.

Another good rule of thumb is if the denominator has x^2+1 in it, it's a good idea to substitute $x = \tan\theta$. This is because the denominator cancels because $\mathrm{d}x = \sec^2\theta \mathrm{d}\theta = (x^2+1)\mathrm{d}\theta$ With the integral (1.11), putting this in, we get

$$\int_0^\infty \frac{\ln x}{x^2+1}\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\tan x)\mathrm{d}x$$

There's two ways worth noting to do this integral. Either use a reflection substitution $u = \frac{\pi}{2} - x$ and use $\tan\left(\frac{\pi}{2} - x\right) = \frac{1}{\tan x}$ to get $I = -I$. Or you can split the integral into a sine and cosine one:

$$\int_0^{\frac{\pi}{2}} \ln(\tan x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} \ln(\sin x)\mathrm{d}x - \int_0^{\frac{\pi}{2}} \ln(\cos x)\mathrm{d}x$$

Now this is 0 because these two functions take the same values on this interval. While they boil down to the same thing, I think it's worth keeping this interpretation in mind.

Another integral which combines all of the above functions is the following

$$I = \int_0^\infty \frac{\arctan(x)}{x(\ln^2(x)+1)} \tag{1.12}$$

With the substitution $u = \frac{1}{x}$ we get

$$I = \int_\infty^0 \frac{\arctan(\frac{1}{u})}{\frac{1}{u}((-\ln(u))^2+1))} - \frac{1}{u^2}\mathrm{d}u = \int_0^\infty \frac{\frac{\pi}{2} - \arctan(u)}{u(\ln^2(u)+1)}\mathrm{d}u = \frac{\pi}{2}\int_0^\infty \frac{\mathrm{d}u}{u(\ln^2(u)+1)} - I$$

Therefore we have

$$2I = \frac{\pi}{2}\int_0^\infty \frac{\mathrm{d}u}{u(\ln^2(u)+1)}$$

Since $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(x)) = \frac{1}{x}$ it makes sense to substitute $v = \ln(u)$. Doing this, the bounds change to $-\infty$ to ∞ and $\mathrm{d}v = \dfrac{\mathrm{d}u}{u}$ so we get

$$2I = \frac{\pi}{2}\int_\infty^\infty \frac{\mathrm{d}v}{v^2+1} = \frac{\pi}{2}\left[\arctan(v)\right]_\infty^\infty = \frac{\pi^2}{2}$$

so $I = \dfrac{\pi^2}{4}$.

1.3 Recurrence relations

For integrals indexed by some number, usually an integer, a recursion relation can often be a good way to reduce it to something which is easier to handle. The first example, which will be relevant later, is

$$I_n = \int_0^\infty x^n e^{-x} \mathrm{d}x \tag{1.13}$$

To form a recursion relation, we can integrate by parts with $u = x^n$ so that $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x}$ so that $v = -e^{-x}$. Then we get

$$I_n = -\left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1}e^{-x}\mathrm{d}x = nI_{n-1}$$

This yields $I_n = n!I_0$ and $I_0 = 1$ can be checked quite easily so we get $I_n = n!$. Note that this integral is defined for non integer values too - something we'll think about later.

Another way to form recurrence relations is to manipulate integrals algebraically, consider

$$I_n = \int_0^1 \frac{x^n}{1+x}\mathrm{d}x$$

for integer n. Adding together I_n and I_{n-1} we get

$$I_n + I_{n-1} = \int_0^1 x^{n-1}\mathrm{d}x = \frac{1}{n} \tag{1.14}$$

Then rearranging this,

$$I_n = \frac{1}{n} - I_{n-1} = -\frac{1}{n(n-1)} + I_{n-2}$$

Repeatedly applying this, if n is odd then it will reduce to I_1 and to I_0 if n is even. We have

$$\begin{aligned} I_0 &= \int_0^1 \frac{1}{1+x}\mathrm{d}x = \ln 2 \\ I_1 &= \int_0^1 \frac{x}{1+x}\mathrm{d}x = \int_0^1 \frac{1+x-1}{1+x}\mathrm{d}x = 1 - \ln 2 \end{aligned}$$

where the second is an example of the 'adding 1 taking away 1' trick which is quite useful. With these, we get

$$I_n = \begin{cases} \ln 2 - \displaystyle\sum_{k=1}^{j} \frac{1}{2k(2k-1)} & n = 2j \\ 1 - \ln 2 - \displaystyle\sum_{k=1}^{j} \frac{1}{2k(2k+1)} & n = 2j+1 \end{cases}$$

Another class of recursion formulas is with powers of trigonometric integrals - integration by parts is used but in combination with some trigonometric identities. Consider

$$I_n = \int_0^{\pi} \sin^n(x)\mathrm{d}x \tag{1.15}$$

Assume $n > 2$ - the reason for this will become clear. Using $\sin^2 x = 1 - \cos^2 x$ we can rewrite the integral as

$$\begin{aligned} I_n &= \int_0^{\pi} \sin^{n-2}(x)(1-\cos^2(x))\mathrm{d}x \\ &= \int_0^{\pi} \sin^{n-2}(x)\mathrm{d}x - \int_0^{\pi} \sin^{n-2}(x)\cos^2(x)\mathrm{d}x \\ &= I_{n-2} - \int_0^{\pi} \sin^{n-2}(x)\cos(x)\cos(x)\mathrm{d}x \end{aligned}$$

Now we can perform integration by parts on this with $u = \cos(x)$ so we have $\frac{\mathrm{d}u}{\mathrm{d}x} = -\sin(x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin^{n-2}(x)\cos(x)$ so that $v = \frac{\sin^{n-1}(x)}{n-1}$.
Then we get

$$\begin{aligned} \int_0^{\pi} \sin^{n-2}(x)\cos(x)\cos(x)\mathrm{d}x &= \left[\tfrac{\cos(x)\sin^{n-1}(x)}{n-1}\right]_0^{\pi} + \frac{1}{n-1}\int_0^{\frac{\pi}{2}} \sin^n(x)\mathrm{d}x \\ &= \frac{1}{n-1} I_n \end{aligned}$$

This then gives us

$$I_n = \frac{n-1}{n} I_{n-2}$$

Repeatedly applying this, we obtain if $n = 2k+1$ is odd

$$\begin{aligned} I_n = \frac{n-1}{n} I_{n-2} = \frac{2k}{2k+1} I_{2k-1} &= \frac{2k(2k-2)}{(2k+1)(2k-1)} I_{2k-3} \\ &= \frac{2k(2k-2)\cdots\times 4\times 2}{(2k+1)(2k-1)\cdots\times 3\times 1} I_1 \\ &= \frac{2^{2k}k^2(k-1)^2(k-2)^2\cdots\times 2^2\times 1^2}{(2k+1)!} I_1 \\ &= \frac{2^{2k}k!^2}{(2k+1)!} I_1 \end{aligned}$$

We can work out that $I_1 = 2$. Similarly for even numbers $n = 2k$ we get

$$I_n = \frac{n-1}{n} I_{n-2} = \frac{2k-1}{2k} I_{2k-2} = \cdots = \frac{(2k)!}{2^{2k}k!^2} I_0$$

We have $I_0 = \pi$ so in summary we get

$$I_n = \begin{cases} \dfrac{\pi(2k)!}{2^{2k}k!^2} & n = 2k \\ \dfrac{2^{2k+1}k!^2}{(2k+1)!} & n = 2k+1 \end{cases}$$

1.4 Wallis Product*

We can use this to calculate the Wallis Product, an interesting product for π.

Using the previous integral (1.15), observing that since $\sin x \leq 1$ for $0 \leq x \leq \pi$, we have

$$\sin^{2n+1}(x) \leq \sin^{2n}(x) \leq \sin^{2n-1}(x) \implies I_{2n+1} \leq I_{2n} \leq I_{2n-1}$$

Dividing through and using the recurrence relation we get

$$1 \leq \frac{I_{2n}}{I_{2n+1}} \leq \frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n}$$

By the squeeze theorem, taking n to ∞

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1 = \frac{\pi}{2} \lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{2k-1}{2k} \cdot \frac{2k+1}{2k} \right)$$

This gives us the Wallis Product

$$\prod_{k=1}^{\infty} \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} = \frac{\pi}{2} \tag{1.16}$$

The Wallis product can be used to get another interesting result. Using the expression derived for I_n directly,

$$\frac{I_{2k+1}}{I_{2k}} = \frac{\left(\frac{2^{2k+1}k!^2}{(2k+1)!}\right)}{\left(\frac{\pi(2k)!}{2^{2k}k!^2}\right)} = \frac{2^{4k+1}k!^4}{\pi(2k)!(2k+1)!} = \left(\frac{2^{2k}k!^2}{(2k)!}\right)^2 \cdot \frac{2}{\pi(2k+1)} \to 1$$

as $k \to \infty$. Rearranging this, we can write it as

$$\frac{1}{2^{2k}} \binom{2k}{k} \sim \frac{1}{\sqrt{\pi k}}$$

This identity can be used in various proofs of Stirling's formula and can actually be made exact by generalising the notion of a factorial! Note that this is an asymptotic expression - it means their ratio tends to 1. As an approximation it also works but only for large k.

1.5 Integral of Inverse

This is a neat trick where spotting it can save you a lot of time. The result is that, for injective functions f:

$$\int_a^b f(x)\mathrm{d}x + \int_{f(a)}^{f(b)} f^{-1}(x)\mathrm{d}x = bf(b) - af(a) \tag{1.17}$$

which can be demonstrated by the following diagram

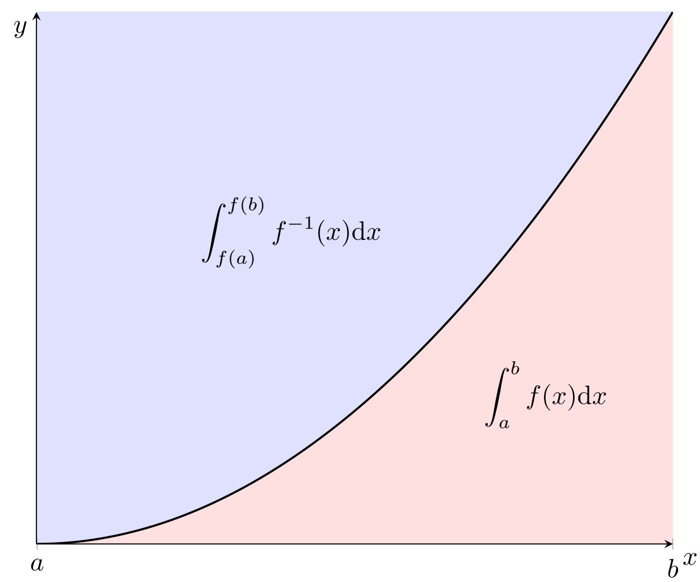

Figure 4: $f(x) = x^2$ with the area under it and its inverse in red and blue.

The result follows from seeing that the two regions together form a rectangle.

A typical integral using this identity is

$$I = \int_0^e W(x)\mathrm{d}x$$

where $W(x)$ is the Lambert W function, the solution to $W(x)e^{W(x)} = x$ i.e the inverse of xe^x. Checking xe^x injective on this interval is a good idea which can be seen by noticing it's a strictly increasing function on this interval. Using (1.17) we get, with $a = 0$ and $b = 1$

$$\int_0^1 xe^x\mathrm{d}x + \int_0^e W(x)\mathrm{d}x = 1 \times e^1 - 0 \times e^0 = e$$

The other integral is doable with integration by parts

$$\int_0^1 xe^x \mathrm{d}x = [xe^x]_0^1 - \int_0^1 e^x \mathrm{d}x$$
$$= e - e + 1 = 1$$

so that we get

$$\int_0^e W(x)\mathrm{d}x = e - 1$$

1.6 The T-Substitution.

When you have complicated trig integrals often a good thing to try is the t-sub - especially if it's a rational function of trig functions. The t sub is also known as the Weierstrass substitution. The goal here is to change a trig integral into a polynomial one which can be solved much easier.

So what is the t-sub? If you have an integral involving trig functions of a variable say, x. The t sub is setting

$$t = \tan\left(\frac{x}{2}\right)$$

Why? Because we can get a nice form for $\mathrm{d}x, \sin x, \cos x, \tan x$ meaning any trig function we have can be turned into a nice (rational) function of t.

So what are these functions then? Let's use double angle formulas to derive them.

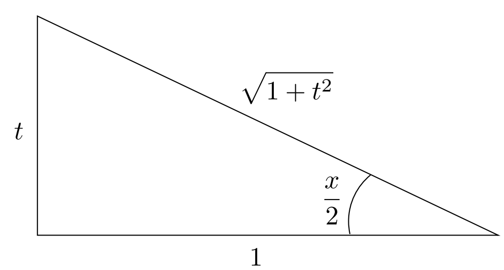

Figure 5: Right angled triangle with $t = \tan\left(\frac{x}{2}\right)$

We see from the diagram above that $\sin\frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$ & $\cos\frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$

Thus we get

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$

Further since $t = \tan\frac{x}{2}$ we get

$$\mathrm{d}t = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)\mathrm{d}x = \frac{1+t^2}{2}\mathrm{d}x$$

Thus we have

$$\int f(\sin x, \cos x, \tan x)\mathrm{d}x = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}\right) \cdot \frac{2}{1+t^2} \cdot \mathrm{d}t$$

This is an example:

$$I = \int \frac{1}{a+\cos x}\,\mathrm{d}x \tag{1.18}$$

where $a > 1$ to ensure that the denominator doesn't blow up. We now substitute $t = \tan\frac{x}{2}$

$$\begin{aligned}
I &= \int \frac{1}{a+\cos x}\,\mathrm{d}x = \int \frac{1}{a+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \cdot \mathrm{d}t \\
&= \int \frac{2}{(a+1)+t^2(a-1)}\mathrm{d}t \\
&= \frac{2}{a+1}\int \frac{1}{1+(t\sqrt{\frac{a-1}{a+1}})^2}\mathrm{d}t \\
&= \frac{2}{\sqrt{a^2-1}}\arctan\left(t\sqrt{\frac{a-1}{a+1}}\right) + C \\
&= \frac{2}{\sqrt{a^2-1}}\arctan\left(\tan\left(\frac{x}{2}\right)\sqrt{\frac{a-1}{a+1}}\right) + C
\end{aligned}$$

1.7 Floor Function Integrals

An interesting, and fairly common class of problems involve the floor function $\lfloor x \rfloor$. These tend to look fairly daunting at the start but there are some standard tricks to decompose them into much more manageable problems. Also Note that this section will be closely linked to the series solution section, for reasons that will hopefully become clear.

For completeness sake the definition of the floor function of x is the greatest integer less than or equal to x denoted by $\lfloor x \rfloor$. For example $\lfloor 2.2 \rfloor = 2$, $\lfloor -2.5 \rfloor = -3$, $\lfloor \pi \rfloor = 3$.

Consider the integral

$$\int_1^\infty \frac{x - \lfloor x \rfloor}{x^4} \mathrm{d}x \tag{1.19}$$

The trick here is to rewrite $\displaystyle\int_1^\infty (\cdots) = \sum_{k=1}^\infty \int_k^{k+1} (\cdots)$ as we know that when $x \in [k, k+1)$ that $\lfloor x \rfloor = k$.

Thus we have:

$$\begin{aligned}
\int_1^\infty \frac{x - \lfloor x \rfloor}{x^4} \mathrm{d}x &= \int_1^\infty \frac{\mathrm{d}x}{x^3} - \sum_{k=1}^\infty \left(\int_k^{k+1} \frac{\lfloor x \rfloor}{x^4} \mathrm{d}x \right) \\
&= \left[\frac{-1}{2x^2} \right]_1^\infty - \sum_{k=1}^\infty \left(\int_k^{k+1} \frac{k}{x^4} \mathrm{d}x \right) \\
&= \left(0 - \left(-\frac{1}{2} \right) \right) - \sum_{k=1}^\infty \left(\left[-\frac{k}{3x^3} \right]_k^{k+1} \right) \\
&= \frac{1}{2} - \sum_{k=1}^\infty \left(\frac{k}{3k^3} - \frac{k+1}{3\left(k+1\right)^3} + \frac{1}{3\left(k+1\right)^3} \right) \\
&\text{The first two terms telescope leaving only } \frac{1}{3} \\
&= \frac{1}{2} - \left(\frac{1}{3} + \sum_{k=1}^\infty \frac{1}{3(k+1)^3} \right) \\
&= \frac{1}{2} - \frac{1}{3} \cdot \zeta(3)
\end{aligned}$$

As you may be able to see the main trick here is splitting the limits of the integral into sections such that the floor function is known. Then we usually get some kind of series which we are able to evaluate or quote.

An example of this is the next integral - I would recommend trying this one yourself before checking the solution.

$$\int_0^\infty e^{-x}\lfloor x\rfloor \mathrm{d}x \tag{1.20}$$

Again we split the integral from 0 to ∞ with a sum.

$$\begin{aligned}
\int_0^\infty e^{-x}\lfloor x\rfloor \mathrm{d}x &= \sum_{k=0}^{\infty}\int_k^{k+1} ke^{-x}\mathrm{d}x \\
&= \sum_{k=0}^{\infty} -k\left[e^{-x}\right]_k^{k+1} \\
&= -1\cdot\sum_{k=0}^{\infty} ke^{-k}\left(\frac{1}{e}-1\right) \\
&= \left(1-\frac{1}{e}\right)\sum_{k=0}^{\infty}\left(-1\cdot\frac{\mathrm{d}}{\mathrm{d}y}e^{-ky}\right)\Bigg|_{y=1} \\
&= \left(1-\frac{1}{e}\right)\cdot(-1)\cdot\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{1-e^{-y}}\right)\Bigg|_{y=1} \\
&= \frac{1}{e-1}
\end{aligned}$$

1.8 Trigonometric Identities

A problem from JEE Main 2016 (I like to have JEE Problems) is

$$I = \int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)\mathrm{d}x \tag{1.21}$$

For this, there doesn't seem to be much which can be done because a substitution will make things nasty outside the arctan. The only thing to do is appeal to identities. A relevant one is the tan addition formula which implies the arctan addition formula (as long as you're careful with the restriction of arctan to the range $[-\frac{\pi}{2},\frac{\pi}{2}]$:

$$\arctan a + \arctan b = \arctan\left(\frac{a+b}{1-ab}\right)$$

Staring at the integrand above, if we want to use this we must have $a+b=1$. The denominator is $1-ab = 1-x(1-x)$ so with $a=x$ and $b=1-x$ we can apply the formula to get

$$\int_0^1 \arctan(x)+\arctan(1-x)\mathrm{d}x = 2\int_0^1 \arctan(x)\mathrm{d}x$$

The use of the addition formula is fine because $\arctan(x)$ and $\arctan(1-x)$ are in the range $[0,\frac{\pi}{4}]$ on the domain $[0,1]$ so their sum is in the range $[0,\frac{\pi}{2}]$.

Now to integrate this, the integral of inverse method could be used. Another way is to use the derivative of inverse formula and integration by parts if it works out nicely, which it does here:

$$\int_0^1 \arctan(x)\mathrm{d}x = [x\arctan(x)]_0^1 - \int_0^1 \frac{x}{x^2+1}\mathrm{d}x = \frac{\pi}{4} - [\tfrac{1}{2}\ln(x^2+1)]_0^1 = \frac{\pi}{4} - \tfrac{1}{2}\ln 2$$

So the original integral is

$$\int_0^1 \arctan\left(\frac{1}{x^2-x+1}\right)\mathrm{d}x = \frac{\pi}{2} - \ln 2$$

Another important thing to keep in mind is using $\sec x$ and $\tan x$ - they have a lot of nice relations together, namely:

$$\sec^2 x = \tan^2 x + 1, \frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x, \frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

It can be quite useful when handling other trigonometric functions like sines and cosines to reduce them to sec and tan. This can be demonstrated by this integral

$$I = \int_0^\pi \frac{x\sin x}{3+\cos^2 x}\mathrm{d}x \tag{1.22}$$

Performing a reflection substitution and adding them together yields

$$2I = \pi\int_0^\pi \frac{\sin x}{3+\cos^2(x)}\mathrm{d}x$$

As it is, I don't really see how to handle it with sines and cosines. However, if you divide through by $\cos^2(x)$ then the integral becomes

$$2I = \pi\int_0^\pi \frac{\sec x\tan x}{3\sec^2(x)+1}\mathrm{d}x$$

Now the substitution $u = \sec x$ works well but we need to be careful with how we go about it as $\sec x$ isn't defined at $x = \frac{\pi}{2}$. So, splitting the integral into two and applying the substitution $u = \sec x$ to both individually we get

$$\begin{aligned}
2I &= \pi\int_0^{\frac{\pi}{2}} \frac{\sec x\tan x}{3\sec^2(x)+1}\mathrm{d}x + \pi\int_{\frac{\pi}{2}}^\pi \frac{\sec x\tan x}{3\sec^2(x)+1}\mathrm{d}x \\
&= \pi\int_1^\infty \frac{\mathrm{d}u}{3u^2+1} + \pi\int_{-\infty}^{-1} \frac{\mathrm{d}u}{3u^2+1} \\
&= 2\pi\int_1^\infty \frac{\mathrm{d}u}{3u^2+1} \\
&= 2\pi\left[\frac{\arctan(\sqrt{3}u)}{\sqrt{3}}\right]_1^\infty = \frac{\pi^2\sqrt{3}}{9}
\end{aligned}$$

Another thing to note is the reflection substitution $u = \pi - x$ can be quite useful for trigonometric functions too as $\cos(\pi - x) = -\cos(x)$ and $\sin(\pi - x) = \sin(x)$ - they stay as the same function but the sign flips for cosines.

A more advanced substitution is $u = \sin x - \cos x = \sqrt{2}\cos\left(x - \frac{3\pi}{4}\right)$. The second form is useful for seeing on what intervals it's injective so we know when it can be used as a substitution. It can be useful when there's at least two of $\sin x, \cos x$ and $\sin(2x)$ in an integral. An example of its use is in the integral

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin(2x)})^2}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{(1+\sqrt{\sin(2x)})^2}\mathrm{d}x \tag{1.23}$$

where the second line is by a reflection substitution. Now substituting $u = \sin x - \cos x$ (it is injective on this interval), we get $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x + \sin x$ taking care of the numerator. For $\sin(2x)$, notice that

$$u^2 = \sin^2 x - 2\sin x\cos x + \cos^2 x = 1 - 2\sin x \cos x \implies \sin(2x) = 1 - u^2$$

Then

$$I = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}u}{(1+\sqrt{1-u^2})^2} = \int_0^1 \frac{\mathrm{d}u}{(1+\sqrt{1-u^2})^2}$$

Now it's in a form where putting a trigonometric sub in is good, with $u = \sin t$ the integral becomes

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x \mathrm{d}x}{(1+\cos x)^2}$$

This is a rational function of cosines so a t sub will work well, turning it into

$$I = \int_0^1 \frac{\frac{1-t^2}{1+t^2}\cdot\frac{2}{1+t^2}}{\left(\frac{2}{1+t^2}\right)^2}\mathrm{d}t = \int_0^1 \frac{1-t^2}{2} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

Another neat trick with trigonometric functions is using Euler's formula, $e^{ix} = \cos x + i\sin x$. This is used a lot in the contour integration section but can be used outside of that. Consider the integral

$$\int_0^{2\pi} \cos^{2n}(x)\mathrm{d}x \tag{1.24}$$

Expanding with Euler's formula,

$$\begin{aligned}\int_0^{2\pi} \cos^{2n} x \mathrm{d}x &= \int_0^{2\pi} \left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n}\mathrm{d}x \\ &= \frac{1}{2^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}\int_0^{2\pi} e^{(2k-2n)ix}\mathrm{d}x\end{aligned}$$

Now for $n \neq k$,

$$\int_0^{2\pi} e^{(2k-2n)ix}\mathrm{d}x = \left[\frac{e^{(2k-2n)ix}}{i(2k-2n)}\right]_0^{2\pi} = 0$$

as $2k-2n$ is an integer. For $n = k$, the integrand is 1 so the integral is 2π which gives us

$$\int_0^{2\pi} \cos^{2n}(x)\mathrm{d}x = \frac{\pi}{2^{2n-1}}\binom{2n}{n}$$

which is a neat way of going about it and a lot nicer than a recurrence method.

1.9 Introducing constants

The title of this section isn't really telling because it covers both the more common adding a constant and subtracting it (usually 1) and the less common multiplying by a constant and dividing it out which isn't often useful. These techniques appear all over the place just like the reflection substitution so there will be lots of examples throughout the book, generally combined with other techniques. A direct example of the technique is

$$I = \int \frac{t^2+7t+8}{t^2+5t+3}\mathrm{d}t = \int \frac{t^2+5t+3}{t^2+5t+3}\mathrm{d}t + \int \frac{2t+5}{t^2+5t+3} = t + \ln(t^2+5t+3) + C$$

Another one which helps us save a lot of work is

$$\int \frac{5\cos x + 7}{2+\cos x}\mathrm{d}x = 5\int \frac{\cos x + 2}{2+\cos x}\mathrm{d}x - \int \frac{3}{2+\cos x}\mathrm{d}x = 5x - 2\sqrt{3}\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + C$$

using the above result. Doing it directly with a t sub would be quite a lot messier - having either a constant or a lone trig function at top is much better.

For an example of multiplying by a constant, a really good example is Problem 21 from the 2021 Integration Bee Round 1 (by coincidence, 2020 and 2021's Problem 21 has been my favourite!)

$$I = \int_0^a \frac{x}{\cos(x)\cos(a-x)}\mathrm{d}x \tag{1.25}$$

The first thing to do is the usual reflection substitution,

$$2I = a\int_0^a \frac{1}{\cos(x)\cos(a-x)}\mathrm{d}x$$

This integral can be done by using product to sum on the denominator but instead a really clever way is to multiply and divide by the constant $\sin a$

$$\begin{aligned}2I = \frac{a}{\sin a}\int_0^a \frac{\sin a}{\cos x\cos(a-x)}\mathrm{d}x &= \frac{a}{\sin a}\int_0^a \frac{\sin x\cos(a-x)+\sin(a-x)\cos x}{\cos x\cos(a-x)}\mathrm{d}x \\ &= \frac{a}{\sin a}\int_0^a \tan x\mathrm{d}x + \int_0^a \tan(a-x)\mathrm{d}x\end{aligned}$$

These two integrals are the same by the reflection substitution and

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\ln|\cos x| = \ln|\sec x|$$

So going back to the integral and dividing by two, we get

$$I = \frac{a \ln|\sec a|}{\sin a}$$

which is really nice!

1.10 Exercises

1. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos 2x + 1)\mathrm{d}x$
2. $\displaystyle\int_0^{2\pi} \sin^{2n} x \mathrm{d}x$
3. $\displaystyle\int_0^{2\pi} \frac{\cos x}{2 + \cos x}\mathrm{d}x$
4. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{2 - \sin(2x)}\mathrm{d}x$
5. $\displaystyle\int_0^{\infty} \frac{\arctan(x)}{1 + x^2}\mathrm{d}x.$
6. $\displaystyle\int_0^{1} \frac{\ln(1 + x)}{1 + x^2}\mathrm{d}x$
7. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2 + 2x + 2}\mathrm{d}x$
8. $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2 + a^2}$
9. $\displaystyle\int_0^{1} \sqrt{\frac{1}{x} - 1}\mathrm{d}x$
10. Calculate $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{x^4 + 1}$ and use this to calculate $\displaystyle\int_0^{1} \sqrt[4]{\frac{1}{x} - 1}\mathrm{d}x.$
11. Show that, for $a > b > 0$, $\displaystyle\int_0^{1} \sqrt[a]{1 - x^b}\mathrm{d}x = \int_0^{1} \sqrt[b]{1 - x^a}\mathrm{d}x$
12. Calculate $\displaystyle\int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}}\mathrm{d}x$

2 Hints on problems

Problem 1

Hint 1: The most similar integral to this so far is the log sine integral (1.9). Maybe it can be manipulated to turn into that or evaluated in a similar way.

Hint 2: Use the double angle formula

Problem 2

Hint 1: This looks quite similar to (1.24)!

Problem 3

Hint 1: This is almost exactly the same as (1.18) - you could use the same technique or you could manipulate the integrand and possibly reuse that result!

Problem 4

Hint 1: It has a $\cos(x)$ and a $\sin(2x)$ together - there's a technique which was covered for situations like this!

Problem 5

Hint 1: Think about a substitution which works nicely with the limits and doesn't change the integrand much.

Problem 6

Hint 1: With a denominator like that, a tan substitution is good.

Hint 2: Try a reflection substitution - you'll need the tan addition formula.

Alternative Hint 2: For an alternative method, use $\tan x = \dfrac{\sin x}{\cos x}$ and see if you can manipulate it to something like the log sine integral.

Problem 7

Hint 1: Complete the square and think about the denominator.

Hint 2: Try something like Problem 6!

Problem 8

Hint 1: This is the sort of integral where it's good to mess around with it for a bit to simplify it.

Hint 2: Try manipulate it into an integral which looks like (1.11)

Problem 9

Hint 1: Try the integral of inverse - it has an inverse as it's monotonic.

Problem 10

Hint 1: Try a reflection substitution and adding the integrals together.

Hint 2: Factor a z^2 from the numerator and denominator.

Hint 3: Complete the square on the denominator.

Problem 11

Hint 1: Calculate the inverse of the integrand.

Problem 12

Hint 1: The denominator and numerator are fairly similar; can you relate them algebraically?

Hint 2: Divide by $n!$ and add and subtract a term on the numerator which is related to the denominator.

3 Solutions

Basic Techniques

Problem 1

Following the hint, using the double angle formula gives us

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos 2x + 1)\mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(2\cos^2 x)\mathrm{d}x = \pi \log 2 + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos x)\mathrm{d}x$$

Now thinking about the graph of $\cos x$, it takes the same values on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as $\sin x$ does on $[0, \pi]$ so we can write

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos 2x + 1)\mathrm{d}x = \pi \log 2 + 2\int_{0}^{\pi} \log(\sin x)\mathrm{d}x = -\pi \log 2$$

Problem 2

The similarity with (1.24) suggests using the same method:

$$\begin{aligned}
\int_0^{2\pi} \sin^{2n} x\mathrm{d}x &= \int_0^{2\pi} \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^{2n} \mathrm{d}x \\
&= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} \int_0^{2\pi} e^{(2k-2n)ix}(-1)^{2n-k}(i)^2 n\mathrm{d}x \\
&= \frac{\pi}{2^{2n-1}} \binom{2n}{n}
\end{aligned}$$

which is exactly the same answer. That makes sense, thinking about the graph of $\sin(x)$ and $\cos(x)$ on the interval $[0, 2\pi]$, they both cover the same values so this could be demonstrated to be the same with some interval splitting and reflection substitutions.

Problem 3

A similar integral done earlier was (1.18). This integral can be manipulated into that form:

$$\int_0^{2\pi} \frac{\cos x}{2 + \cos x}\mathrm{d}x = \int_0^{2\pi} 1 - \frac{2}{2 + \cos x}\mathrm{d}x = 2\pi - 2\int_0^{2\pi} \frac{\mathrm{d}x}{2 + \cos x}$$

Now we know that

$$\int \frac{\mathrm{d}x}{2 + \cos x} = \frac{2}{\sqrt{3}} \arctan\left(\sqrt{\frac{1}{3}} \tan\left(\frac{x}{2}\right)\right) + C$$

We can't substitute the limits 0 to 2π in though - this integral was worked out with a t sub so we need to make sure that it's done on an injective domain so splitting into $[0, \pi]$, we get

$$\int_0^{2\pi} \frac{\mathrm{d}x}{2+\cos x} = \int_0^{\pi} \frac{\mathrm{d}x}{2+\cos x} + \int_{\pi}^{2\pi} \frac{\mathrm{d}x}{2+\cos x} = 2\int_0^{\pi} \frac{\mathrm{d}x}{2+\cos x} = \frac{2\pi}{\sqrt{3}}$$

Putting this all together, we get

$$\int_0^{2\pi} \frac{\cos x}{2+\cos x}\mathrm{d}x = 2\pi\left(1 - \frac{2}{\sqrt{3}}\right)$$

Problem 4

This integral has similar functions involved as (1.23) so a similar approach could work. Using a reflection substitution to put it in the right form,

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{2-\sin 2x}\mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{2-\sin 2x}\mathrm{d}x$$

Now using the substitution $u = \sin x - \cos x$ we get

$$I = \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}u}{1+u^2} = \frac{1}{2}\left[\arctan(u)\right]_{-1}^{1} = \frac{\pi}{4}$$

Problem 5

This integrand is full of things which work nicely with a $u = \dfrac{1}{x}$ substitution; the $[0, \infty]$ limits, $\arctan(x)$ and the $\frac{1}{1+x^2}$. Going ahead and using that substitution, we get

$$I = \int_0^{\infty} \frac{\arctan(x)}{1+x^2}\mathrm{d}x = \int_0^{\infty} \frac{\arctan(\frac{1}{x})}{1+x^2}\mathrm{d}x = \int_0^{\infty} \frac{\frac{\pi}{2} - \arctan(x)}{1+x^2}\mathrm{d}x = \frac{\pi^2}{4} - I$$

Rearranging this, we get

$$I = \frac{\pi^2}{8}$$

Problem 6

This integral is famous - it's known as Serret's integral. Following the hint, using a tan substitution, we get

$$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\mathrm{d}x$$

Now with a reflection substitution, this becomes

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan(\tfrac{\pi}{4} - x))\mathrm{d}x$$

With the addition formula for tan, we have

$$\tan(\tfrac{\pi}{4} - x) = \frac{\tan(\frac{\pi}{4}) - \tan(x)}{1 + \tan(\frac{\pi}{4})\tan(x)} = \frac{1 - \tan(x)}{1 + \tan(x)}$$

Putting this into the above,

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan(\tfrac{\pi}{4} - x))\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan(x)}{1 + \tan(x)}\right)\mathrm{d}x = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan(x)}\right) = \frac{\pi}{4}\ln 2 - I$$

This gives us

$$\int_0^1 \frac{\ln(1 + x)}{1 + x^2}\mathrm{d}x = \frac{\pi}{8}\ln 2$$

For the alternative method (which I prefer - seems more natural), writing the tan integral in terms of sines and cosines

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\mathrm{d}x &= \int_0^{\frac{\pi}{4}} \ln(\sin x + \cos x)\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(\cos x)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{4}} \ln(\sqrt{2}\sin\left(x - \frac{\pi}{4}\right))\mathrm{d}x - \int_0^{\frac{\pi}{4}} \ln(\cos x)\mathrm{d}x \\
&= \frac{\pi}{8}\ln 2 + \int_0^{\frac{\pi}{4}} \ln(\sin\left(x - \frac{\pi}{4}\right)) - \int_0^{\frac{\pi}{4}} \ln(\cos x)\mathrm{d}x = \frac{\pi}{8}\ln 2
\end{aligned}$$

where the second to last line can be seen by a reflection substitution or just thinking about the values sine and cosine take on those intervals.

Problem 7

This one can be treated like Problem 6. Completing the square on the denominator gives us

$$I = \int_0^{\infty} \frac{\ln x}{(x + 1)^2 + 1}\mathrm{d}x$$

In this form, a tan substitution is good; $x + 1 = \tan\theta$ which gives us

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\tan\theta - 1)\mathrm{d}\theta$$

This can now be treated like the alternative method of Problem 6.

$$\begin{aligned} I &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin\theta - \cos\theta)\mathrm{d}\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cos\theta)\mathrm{d}\theta \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right))\mathrm{d}\theta - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\cos\theta)\mathrm{d}\theta = \frac{\pi}{8}\ln 2 \end{aligned}$$

which is the same answer as the previous problem! I tried to manipulate them to see if there's a way to see if they're equal but couldn't do it.

Problem 8

There are two approaches to this - we could go straight with $x = a\tan u$ or just manipulate it to turn it into an integral we've done before - it looks similar to (1.11). To do that, putting in $x = au$ gives

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\mathrm{d}x = \frac{1}{a}\int_0^\infty \frac{\ln(au)}{u^2+1}\mathrm{d}u = \frac{1}{a}\left(\int_0^\infty \frac{\ln(a)}{u^2+1}\mathrm{d}u + \int_0^\infty \frac{\ln(u)}{u^2+1}\mathrm{d}u\right)$$

The second integral we've done before; its 0 and the first integral is fairly standard, giving us

$$\int_0^\infty \frac{\ln x}{x^2+a^2}\mathrm{d}x = \frac{\pi\ln a}{2a}$$

Problem 9

Following the hint for this problem, the inverse of the function is $\frac{1}{x^2+1}$ whose integral isn't too difficult. When $\frac{1}{x^2+1} = 0$ is 1, $x = 0$ but for 0, it's $x \to \infty$. However substituting this into the formula could cause problems so taking limits instead,

$$\int_0^b \frac{\mathrm{d}x}{x^2+1} - \int_{\frac{1}{b^2+1}}^1 \sqrt{\frac{1}{x} - 1}\mathrm{d}x = \frac{b}{b^2+1}$$

Taking the limit as $b \to \infty$ we get

$$\int_0^1 \sqrt{\frac{1}{x} - 1}\mathrm{d}x = \int_0^\infty \frac{\mathrm{d}x}{x^2+1} = \frac{\pi}{2}$$

Problem 10

This integral is fairly tough to do with these methods, it can be done with more advanced techniques like contour integration but this method is much nicer. First by using a reflection substitution, we get

$$I = \int_0^\infty \frac{\mathrm{d}x}{x^4+1} = \int_\infty^0 \frac{-\frac{\mathrm{d}x}{x^2}}{\frac{1}{x^4}+1} = \int_0^\infty \frac{x^2\mathrm{d}x}{x^4+1}$$

Adding these two together, we get

$$2I = \int_0^\infty \frac{x^2+1}{x^4+1}\mathrm{d}x$$

Now the difficult step is to divide by x^2, then completing the square on this gives us

$$I = \frac{1}{2}\int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x = \frac{1}{2}\left(\int_0^\infty \frac{1+\frac{1}{x^2}}{(x-\frac{1}{x})^2+2}\mathrm{d}x\right)$$

The numerator is now actually the derivative of $x - \frac{1}{x}$ so substituting $u = x - \frac{1}{x}$ (which is injective as it's always increasing; the derivative of u is always positive) gives us

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{\mathrm{d}u}{u^2+2} = \frac{\pi}{2\sqrt{2}}$$

by standard tan substitution stuff.

Another 'basic technique' method would be to use partial fractions but I really don't like partial fractions and the factorisation of x^4+1 isn't that easy to find; once you know it exists it's a bit easier though!

For the other part, use the integral of inverse, it works out the same as Problem 9.

Problem 11

Following the hint, calculating the inverse of the integrand,

$$y = \sqrt[a]{1-x^b} \implies x^b = 1-y^a \implies x = \sqrt[b]{1-x^a}$$

Now using the integral of inverse formula, with $f(x) = \sqrt[a]{1-x^b}$, $f(1) = 0$ and $f(0) = 1$ so

$$\int_0^1 \sqrt[a]{1-x^b}\mathrm{d}x + \int_1^0 \sqrt[b]{1-x^a}\mathrm{d}x = 0 \implies \int_0^1 \sqrt[a]{1-x^b}\mathrm{d}x = \int_0^1 \sqrt[b]{1-x^a}$$

As for the actual value of the integral, that can be done with something called the Beta Function which will be in one of the later sections!

Problem 12

With indefinite integrals, there's much less you can actually do - it'll come down to some sort of algebra tricks/simplification, by parts, a substitution or recognising the integrand is a derivative most of the time. In this case, it's a few of these together. The similarity in the denominator and numerator suggests doing this:

$$
\begin{aligned}
I = \int \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} \mathrm{d}x &= n! \int \frac{\left(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right) - \left(\frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right)}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} \mathrm{d}x \\
&= n! x - n! \int \frac{\left(\frac{x^{n-1}}{(n-1)!} + \cdots + x + 1 \right)}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} \mathrm{d}x \\
&= n! x - n! \ln \left(1 + x + \cdots + \frac{x^n}{n!} \right) + C
\end{aligned}
$$

This integral can be used for some theoretical purposes too. If you're trying to prove that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ then writing

$$
\int_0^t \frac{\frac{x^n}{n!}}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}} \mathrm{d}x = t - \ln \left(1 + t + \cdots + \frac{t^n}{n!} \right)
$$

The integrand is upper bounded by $\frac{x^n}{n!}$ by replacing the denominator with 1 so the integral is upper bounded by $\dfrac{1}{n \cdot n!}$. So taking $n \to \infty$, we have

$$
\lim_{n \to \infty} t - \ln \left(1 + t + \cdots + \frac{t^n}{n!} \right) = 0 \implies e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}
$$

Of course this requires you to know that $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$ so that you can derive things such as the derivative of $\ln x$ etc.